What are degrees of freedom?

As we were teaching a multivariate statistics course for doctoral students, one of the students in the class asked, "What are degrees of freedom? I know it is not good to lose degrees of freedom, but what are they?" Other students in the class waited for a clear-cut response. As we tried to give a textbook answer, we were not satisfied and we did not get the sense that our students understood. We looked through our statistics books to determine whether we could find a more clear way to explain this term to social work students. The wide variety of language used to define degrees of freedom is enough to confuse any social worker! Definitions range from the broad, "Degrees of freedom are the number of values in a distribution that are free to vary for any particular statistic" (Healey, 1990, p. 214), to the technical:

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Statisticians start with the number of terms in
the sum [of squares], then subtract the number
of mean values that were calculated along the
way. The result is called the degrees of freedom,
for reasons that reside, believe it or not, in the
theory of thermodynamics. (Norman & Streiner,
2003, p. 43)
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Authors who have tried to be more specific have defined degrees of freedom in relation to sample size (Trochim, 2005; Weinbach & Grinnell, 2004), cell size (Salkind, 2004), the number of relationships in the data (Walker, 1940), and the difference in dimensionalities of the parameter spaces (Good, 1973). The most common definition includes the number or pieces of information that are free to vary (Healey, 1990; Jaccard & Becker, 1990; Pagano, 2004; Warner, 2008; Wonnacott & Wonnacott, 1990). These specifications do not seem to augment students' understanding of this term. Hence, degrees of freedom are conceptually difficult but are important to report to understand statistical analysis. For example, without degrees of freedom, we are unable to calculate or to understand any underlying population variability. Also, in a bivariate and multivariate analysis, degrees of freedom are a function of sample size, number of variables, and number of parameters to be estimated; therefore, degrees of freedom are also associated with statistical power. This research note is intended to comprehensively define degrees of freedom, to explain how they are calculated, and to give examples of the different types of degrees of freedom in some commonly used analyses.

DEGREES OF FREEDOM DEFINED

In any statistical analysis the goal is to understand how the variables (or parameters to be estimated) and observations are linked. Hence, degrees of freedom are a function of both sample size (N) (Trochim, 2005) and the number of independent variables (k) in one's model (Toothaker & Miller, 1996; Walker, 1940; Yu, 1997). The degrees of freedom are equal to the number of independent observations (N), or the number of subjects in the data, minus the number of parameters (k) estimated (Toothaker & Miller, 1996; Walker, 1940). A parameter (for example, slope) to be estimated is related to the value of an independent variable and included in a statistical equation (an additional parameter is estimated for an intercept in a general linear

model). A researcher may estimate parameters using different amounts or pieces of information, and the number of independent pieces of information he or she uses to estimate a statistic or a parameter are called the degrees of freedom (df) (HyperStat Online, n.d.). For example, a researcher records income of N number of individuals from a community. Here he or she has Nindependent pieces of information (that is, N points of incomes) and one variable called income (k);in subsequent analysis of this data set, degrees of freedom are associated with both N and k. For instance, if this researcher wants to calculate sample variance to understand the extent to which incomes vary in this community, the degrees of freedom equal N - k. The relationship between sample size and degrees of freedom is positive; as sample size increases so do the degrees of freedom. On the other hand, the relationship between the degrees of freedom and number of parameters to be estimated is negative. In other words, the degrees of freedom decrease as the number of parameters to be estimated increases. That is why some statisticians define degrees of freedom as the number of independent values that are left after the researcher has applied all the restrictions (Rosenthal, 2001; Runyon & Haber, 1991); therefore, degrees of freedom vary from one statistical test to another (Salkind, 2004). For the purpose of clarification, let us look at some examples.

A Single Observation with One Parameter to Be Estimated

If a researcher has measured income (k = 1) for one observation (N = 1) from a community, the mean sample income is the same as the value of this observation. With this value, the researcher has some idea of the mean income of this community but does not know anything about the population spread or variability (Wonnacott & Wonnacott, 1990). Also, the researcher has only one independent observation (income) with a parameter that he or she needs to estimate. The degrees of freedom here are equal to N - k. Thus, there is no degree of freedom in this example (1 - 1 = 0). In other words, the data point has no freedom to vary, and the analysis is limited to the presentation of the value of this data point (Wonnacott & Wonnacott, 1990; Yu, 1997). For us to understand data variability, N must be larger than 1.

Multiple Observations (N) with One Parameter to Be Estimated

Suppose there are N observations for income. To examine the variability in income, we need to estimate only one parameter (that is, sample variance) for income (k), leaving the degrees of freedom of N - k. Because we know that we have only one parameter to estimate, we may say that we have a total of N - 1 degrees of freedom. Therefore, all univariate sample characteristics that are computed with the sum of squares including the standard deviation and variance have N - 1 degrees of freedom (Warner, 2008).

Degrees of freedom vary from one statistical test to another as we move from univariate to bivariate and multivariate statistical analysis, depending on the nature of restrictions applied even when sample size remains unchanged. In the examples that follow, we explain how degrees of freedom are calculated in some of the commonly used bivariate and multivariate analyses.

Two Samples with One Parameter (or t Test)

If the assumption of equal variance is violated and the two groups have different variances as is the case in this example, where the folded F test or Levene's F weighted statistic is significant, indicating that the two groups have significantly different variances, the value for degrees of freedom (100) is no longer accurate. Therefore, we need to estimate the correct degrees of freedom (SAS Institute, 1985; also see Satterthwaite, 1946, for the computations involved in this estimation).

We can estimate the degrees of freedom according to Satterthwaite's (1946) method by using the following formula:

df Satterthwaite =

[MATHEMATICAL EXPRESSION NOT REPRODUCIBLE IN ASCII]

where $[n.sub.1]$ = sample size of group 1, $[n.sub.2]$ = sample size of group 2, and $[S.sub.1]$ and $[S.sub.2]$ are the standard deviations of groups 1 and 2, respectively. By inserting subgroup data from Table 1, we arrive at the more accurate degrees of freedom as follows:

[MATHEMATICAL EXPRESSION NOT REPRODUCIBLE IN ASCII]

Because the assumption of equality of variances is violated, in the previous analysis the Satterthwaite's value for degrees of freedom, 96.97 (SAS rounds it to 97), is accurate, and our earlier value, 100, is not. Fortunately, it is no longer necessary to hand calculate this as major statistical packages such as SAS and SPSS provide the correct value for degrees of freedom when the assumption of equal variance is violated and equal variances are not assumed. This is the fourth value for degrees of freedom in our example, which appears in Table 1 as 97 in SAS and 96.967 in SPSS. Again, this value is the correct number to report, as the assumption of equal variances is violated in our example.

Comparing the Means of g Groups with One Parameter (Analysis of Variance)

* The second type of degrees of freedom, called the within-groups degrees of freedom or error degrees of freedom, is derived from subtracting the model degrees of freedom from the corrected total degrees of freedom. The within-groups degrees of freedom equal the total number of observations minus the number of groups to be compared, $[n.sub.1]$ ... $[n.sub.g]$ - g This value also accounts for the denominator degrees of freedom for calculating the F statistic in an ANOVA.

* Calculating the third type of degrees of freedom is straightforward. We know that the sum of deviation from the mean or $[summation]([Y.sub.i] - [bar.Y])$ = 0.We also know that the total sum of squares or $[summation][([Y.sub.i] - [bar.Y]).sup.2]$ is nothing but the sum of $[N.sup.2]$ deviations from the mean. Therefore, to estimate the total sum of squares $[summation][([Y.sub.i] -[bar.Y]).sup.2]$, we need only the sum of N - 1 deviations from the mean. Therefore, with the total sample size we can obtain the total degrees of freedom, or corrected total degrees of freedom, by using the formula N- 1.

In Table 2, we show the SAS and SPSS output with these three different values of degrees of freedom using the ANOVA procedure. The dependent variable, literacy rate, is continuous, and the independent variable, political freedom or FREEDOMX, is nominal. Countries are classified into three groups on the basis of the amount of political freedom each country enjoys: Countries that enjoy high political freedom are coded as 1 (n = 32), countries that enjoy moderate political freedom are coded as 2 (n = 34), and countries that enjoy no political freedom are coded as 3 (n = 36). The mean literacy rates (dependent variable) of these groups of countries are examined. The null hypothesis tests the assumption that there is no significant difference in the literacy rates of these countries according to their level of political freedom.

The first of the three degrees of freedom, the between-groups degrees of freedom, equals g - 1. Because there are three groups of countries in this analysis, we have 3 - 1 = 2 degrees of freedom. This accounts for the numerator degrees of freedom in estimating the F statistic. Second, the within-groups degrees of freedom, which accounts for the denominator degrees of freedom for calculating the F statistic in ANOVA, equals [n.sub.1] ... [n.sub.g] - g. These degrees of freedom are calculated as 32 34 36 - 3 = 99. Finally, the third degrees of freedom, the total degrees of freedom, are calculated as N - 1 (102 - 1 = 101). When reporting F values and their respective degrees of freedom, researchers should report them as follows: The independent and the dependent variables are significantly related [$F(2, 99) = 16.64, p < .0001$].

Degrees of Freedom in Multiple Regression Analysis

We skip to multiple regression because degrees of freedom are the same in ANOVA and in simple regression. In multiple regression analysis, there is more than one independent variable and one dependent variable. Here, a parameter stands for the relationship between a dependent variable (Y) and each independent variable (X). One must understand four different types of degrees of freedom in multiple regression.

* The first type is the model (regression) degrees of freedom. Model degrees of freedom are associated with the number of independent variables in the model and can be understood as follows:

A null model or a model without independent variables will have zero parameters to be estimated. Therefore, predicted Y is equal to the mean of Y and the degrees of freedom equal 0.

A model with one independent variable has one predictor or one piece of useful information (k = 1) for estimation of variability in Y. This model must also estimate the point where the regression line originates or an intercept. Hence, in a model with one predictor, there are (k 1) parameters--k regression coefficients plus an intercept--to be estimated, with k signifying the number of predictors. Therefore, there are [(k 1) - 1], or k degrees of freedom for testing this regression model.

Accordingly, a multiple regression model with more than one independent variable has some more useful information in estimating the variability in the dependent variable, and the model degrees of freedom increase as the number of independent variables increase. The null hypothesis is that all of the predictors have the same regression coefficient of zero, thus there is

only one common coefficient to be estimated (Dallal, 2003). The alternative hypothesis is that the regression coefficients are not zero and that each variable explains a different amount of variance in the dependent variable. Thus, the researcher must estimate k coefficients plus the intercept. Therefore, there are (k 1) - 1 or k degrees of freedom for testing the null hypothesis (Dallal, 2003). In other words, the model degrees of freedom equal the number of useful pieces of information available for estimation of variability in the dependent variable.

* The second type is the residual, or error, degrees of freedom. Residual degrees of freedom in multiple regression involve information of both sample size and predictor variables. In addition, we also need to account for the intercept. For example, if our sample size equals N, we need to estimate k 1 parameters, or one regression coefficient for each of the predictor variables (k) plus one for the intercept. The residual degrees of freedom are calculated N- (k 1). This is the same as the formula for the error, or within-groups, degrees of freedom in the ANOVA. It is important to note that increasing the number of predictor variables has implications for the residual degrees of freedom. Each additional parameter to be estimated costs one residual degree of freedom (Dallal, 2003). The remaining residual degrees of freedom are used to estimate variability in the dependent variable.

* The third type of degrees of freedom is the total, or corrected total, degrees of freedom. As in ANOVA, this is calculated N - 1.

* Finally, the fourth type of degrees of freedom that SAS (and not SPSS) reports under the parameter estimate in multiple regression is worth mentioning. Here, the null hypothesis is that there is no relationship between each independent variable and the dependent variable. The degree of freedom is always 1 for each relationship and therefore, some statistical software, such as SPSS, do not bother to report it.

In the example of multiple regression analysis (see Table 3), there are four different values of degrees of freedom. The first is the regression degrees of freedom. This is estimated as (k 1) - 1 or (6 1) - 1 = 6, where k is the number of independent variables in the model. Second, the residual degrees of freedom are estimated as N - (k 1). Its value here is 99 - (6 1) = 92. Third, the total degrees of freedom are calculated N - 1 (or 99 -1 = 98). Finally, the degrees of freedom shown under parameter estimates for each parameter always equal 1, as explained above. F values and the respective degrees of freedom from the current regression output should be reported as follows: The regression model is statistically significant with $F(6, 92) = 44.86, p < .0001$.

Degrees of Freedom in a Nonparametric Test

Pearson's chi square, or simply the chi-square statistic, is an example of a nonparametric test that is widely used to examine the association between two nominal level variables. According to Weiss (1968) "the number of degrees of freedom to be associated with a chi-square statistic is equal to the number of independent components that entered into its calculation" (p. 262). He further explained that each cell in a chi-square statistic represents a single component and that an independent component is one where neither observed nor expected values are determined by the frequencies in other cells. In other words, in a contingency table, one row and one column are

fixed and the remaining cells are independent and are free to vary. Therefore, the chi-square distribution has (r - 1) x (c - 1) degrees of freedom, where r is the number of rows and c is the number of columns in the analysis (Cohen, 1988; Walker, 1940; Weiss, 1968). We subtract one from both the number of rows and columns simply because by knowing the values in other cells we can tell the values in the last cells for both rows and columns; therefore, these last cells are not independent.

Readers may note that there are three values under degrees of freedom in Table 4. The first two values are calculated the same way as discussed earlier and have the same values and are reported most widely. These are the values associated with the Pearson chi-square and likelihood ratio chi-square tests. The final test is rarely used. We explain this briefly. The degree of freedom for the Mantel--Haenszel chisquare statistic is calculated to test the hypothesis that the relationship between two variables (row and column variables) is linear; it is calculated as (N- 1) x [r.sup.2], where [r.sup.2] is the Pearson product-moment correlation between the row variable and the column variable (SAS Institute, 1990). This degree of freedom is always I and is useful only when both row and column variables are ordinal.

CONCLUSION

Yu (1997) noted that "degree of freedom is an intimate stranger to statistics students" (p. 1). This research note has attempted to decrease the strangeness of this relationship with an introduction to the logic of the use of degrees of freedom to correctly interpret statistical results. More advanced researchers, however, will note that the information provided in this article is limited and fairly elementary. As degrees of freedom vary by statistical test (Salkind, 2004), space prohibits a more comprehensive demonstration. Anyone with a desire to learn more about degrees of freedom in statistical calculations is encouraged to consult more detailed resources, such as Good (1973), Walker (1940), and Yu (1997).

Finally, for illustrative purposes we used World Data that reports information at country level. In our analysis, we have treated each country as an independent unit of analysis. Also, in the analysis, each country is given the same weight irrespective of its population size or area. We have ignored limitations that are inherent in the use of such data. We warn readers to ignore the statistical findings of our analysis and take away only the discussion that pertains to degrees of freedom.

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